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UNIQUE ERGODICITY OF SOME FLOWS RELATED TO AXIOM A DIFFEOMORPHISMS

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ABSTRACT

Continuous flows, whose orbits are the unstable manifolds of certain Axiom A attractors, are shown to be uniquely ergodic. The approach used is symbolic dynamics. Equicontinuity (and lack of it) for these flows is also discussed.

It is a result of H. Furstenberg [8] that the classical horocycle flow is uniquely ergodic. This means that the flow has a unique invariant Borel probability measure. Now the horocycle flow is related to the geodesic flow of a compact two dimensional Riemannian manifold of constant negative curvature in the following way: any two points x and y on the same horocycle orbit are backwards asymptotic under the geodesic flow $\{\phi_t\}$ (i.e., $\lim_{t\to-\infty}$ dist. $(\phi_t x, \phi_t y) = 0$). In the language of dynamical systems then, the horocycle orbits are the unstable manifolds for the geodesic flow.

In light of this, C. Pugh suggested that Furstenberg's result might be generalized to the variable negative curvature case or perhaps still more generally to the Axiom A case (the geodesic flow is a special type of Axiom A flow). That is, suppose that one defines a continuous flow (called the W^u flow) whose orbits are the unstable manifolds of a basic set for an Axiom A flow (see [16]). Is this flow uniquely ergodic? First, to even pose this question one must assume that the unstable manifolds are one-dimensional and that the foliation of unstable manifolds (called the W^u foliation) on the basic set is orientable: for a flow automatically orients its orbits. Then one must rule out some immediate counterexamples (constant time suspension of Axiom A diffeomorphisms) and would probably want to assume that the basic set is an attractor. Under these restrictions we believe that the flow is uniquely ergodic,

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but we are unable to prove this at present. This would generalize Furstenberg's result.

The purpose of this paper is to prove the analogue of the above conjecture for an attractor of an Axiom A diffeomorphism (as opposed to an Axiom A flow). These have W^u foliations just as Axiom A flows do. Our main result is (see Section 2 for definitions):

THEOREM 2.6. *W^u flows on a connected attractor for an Axiom A diffeomorphism f are uniquely ergodic.*

Again, implicit in the statement is that the W^u foliation is one-dimensional and orientable. However, neither of these assumptions seems to be particularly relevant. One might be able to circumvent these assumptions by using a notion of transverse measure for foliations (See [14], [15]). In this context the analogue of our theorem would be that there is a unique (up to constant multiple) transverse measure for the W^u foliation on a connected Axiom A attractor. This can probably be proved by viewing our approach in terms of cross-sections and suspensions.

The measure which maximizes entropy for f (see [14]) will be the unique invariant measure for the W^u flow, provided one has the right parametrization: namely, one parametrizes the orbits by the measures on unstable manifolds constructed in [14], [15].

Our approach was inspired by Williams' paper [20]. The idea is to apply a version of the Perron-Frobenius theorem to the partition matrix for a Markov partition, a certain finite cover of closed "rectangles" $\{A_i\}$ which meet only on their boundaries: one can express the unstable manifold of a point x as a countable union of consecutive subarcs $\{W_j(x)\}$, meeting only at endpoints, such that each $W_j(x)$ crosses once through a rectangle $A_{b_j(x)}$. Now

$$f^n W_q(f^{-n}x) = \bigcup_{j=k_q}^{k_q'-1} W_j(x)$$

and if n is large enough, then for all i and q, the proportion of i's that appear in $\{b_j(x)\}$, $j = k_q, \cdots k_q' - 1$, will be approximately the same for all x (by the Perron-Frobenius theorem). This means that the unstable manifolds are all distributed through the Markov partition in approximately the same way. Since the Markov partition generates (under the action of f), this will imply that the orbits of the W^u flow are all distributed throughout the space in the same way, and this will yield unique ergodicity.

In Section 1 we place the problem in the setting of time averages. Section 2 provides some necessary background on Axiom A diffeomorphisms. In Section 3 we prove the main result. The last section is devoted to a discussion of a related property: equicontinuity. We show that some orientable, one-dimensional W^u foliations (those coming from Anosov diffeomorphisms) admit equicontinuous flows, but others don't.

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1. Unique ergodicity in terms of time averages

Let X be a compact metric space and $\{\psi_t : t \in \mathbf{R}\}$ a continuous flow (i.e., each $\psi_t : X \to X$ is a homeomorphism, $\psi_{t+s} = \psi_t \circ \psi_s$ and the map $(t,x) \to \psi_t(x)$ is continuous). The orbit of a point x is $\{\psi_t(x) : t \in \mathbf{R}\}$ and the positive semi-orbit is $\{\psi_t(x) : t \geqq 0\}$. A Borel probability measure is a measure on the Borel subsets of X such that $\mu(X) = 1$. It is called invariant (for ψ) if $\mu(A) = \mu(\psi_t A)$ for all Borel sets A and $t \in \mathbf{R}$. The flow $\{\psi_t\}$ is called uniquely ergodic if it has exactly one invariant Borel probability measure.

Our viewpoint is based on the following fact due to Kryloff and Bogoliouboff [10].

TIME AVERAGE CRITERION 1.1. *$\{\psi_t\}$ is uniquely ergodic if and only if for all continuous functions $h : X \to \mathbf{R}$,*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T h \circ \psi_t(x)\, dt$$

exists and is constant (i.e. independent of x).

In the following lemma we see that it really is not necessary to sample the time averages at all times T, nor at all points x. In fact it is sufficient to sample only every so often over just a dense set, provided that you sample regularly:

LEMMA 1.2. *Let $h : X \to \mathbf{R}$ be continuous. Then*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T h \circ \psi_t(x)\, dt$$

exists and is constant provided for all $\varepsilon > 0$, there exist $M_\varepsilon > 0$, $r_\varepsilon > 0$ and a dense set $X_\varepsilon \subset X$ such that for each $x \in X_\varepsilon$ there is an unbounded strictly increasing sequence $\{S_m(x) : m \geqq 0\}$ satisfying

(a) $S_0(x) \leqq 0 < S_1(x)$ *and for each* $m \geqq 0$, $S_{m+1}(x) - S_m(x) \leqq M_\varepsilon$,

(b) *for each* $m \geqq 0$, $\left| \frac{1}{S_m(x) - S_0(x)} \left(\int_{S_0(x)}^{S_m(x)} h \circ \psi_t(x) dt \right) - r_\varepsilon \right| \leqq \varepsilon$.

PROOF. Letting $\varepsilon > 0$, choose $M' > M_\varepsilon$ such that $(\|h\| \cdot M_\varepsilon)/M' < \varepsilon$. Let $T \geqq M'$. For $x \in X_\varepsilon$, choose $m = m(T, x)$, the largest integer such that $S_m(x) \leqq T$. Then one can check (using (a) and the choice of M') that

$$\left| \frac{1}{T} \left(\int_0^T h \circ \psi_t(x) dt \right) - \frac{1}{S_m(x) - S_0(x)} \left(\int_{S_0(x)}^{S_m(x)} h \circ \psi_t(x) dt \right) \right| \leqq 4\varepsilon .$$

Using this and (b) we then have

$$\left| \frac{1}{T} \left(\int_0^T h \circ \psi_t(x) dt \right) - r_\varepsilon \right| \leqq 5\varepsilon .$$

Since X_ε is dense, the same estimate holds for all $x \in X$. This yields the lemma. ■

We now discretize the process. The idea is to chop up the space X into small boxes and then, over a finite time interval, count the number of times that a given orbit passes through each box. This will give a relative frequency of visits to each box. From these frequencies we can compute time averages of continuous functions. To do this we need a good set of boxes. Roughly, the boxes must be small, look like "flowboxes" and for some dense set X_* the relative frequency with which the positive semi-orbit of x visits a given box is approximately independent of $x \in X_*$.

DEFINITION 1.3. A finite collection $\mathscr{A} = \{A_i : i \in \Lambda\}$ of subsets of X is δ-*good* (for the flow ψ) if (I), (II), and (III) hold:

(I) $\mathscr{A}$ covers X and each diam $A_i < \delta$.

There is a ψ-invariant, dense set X_* such that

(II) (a) For $x \in X_*$ both

$$B_\pm(x) = \left\{ \begin{matrix} t \geqq 0 \\ t \leqq 0 \end{matrix} : \psi_t(x) \text{ belongs to more than one } A_i \right\}$$

are discrete and non-empty (whence unbounded by ψ-invariance of X_*). Labelling the elements of $B_+(x) \cup B_-(x)$:

$$\cdots < T_{-2}(x) < T_{-1}(x) < T_0(x) \leqq 0 < T_1(x) < T_2(x) < \cdots$$

(b) for each integer j, there is $b_j(x)$ such that

$$\{\psi_t(x) : t \in [T_j(x),\ T_{j+1}(x)]\} \subset A_{b_j(x)} .$$

(III) For $i \in \Lambda$ there exist $r_i \geqq 0$ with $\Sigma_{i\in\Lambda} r_i = 1$ and $M_* \geqq 1$ such that if $x \in X_*$ there is an increasing sequence of integers $\{K_m(x): m \geqq 0\}$ satisfying

(a) $K_0(x) \leqq 0 < K_1(x)$ and for each $m \geqq 0$ $\quad K_{m+1}(x) - K_m(x) \leqq M_*$,

(b) for each $m \geqq 0$ $\quad |\eta_{im}(x) - r_i| < \delta r_i$

where

$$\eta_{im}(x) = \frac{\#\{j \in [K_0(x),\ K_m(x)-1]: b_j(x) = i\}}{K_m(x) - K_0(x)}.$$

We remark that (II) is required mainly for the purpose of formulating (III). The quantity in (III), $\eta_{im}(x)$, represents the relative frequency of visits to A_i over the time interval $[T_{K_0(x)}(x),\ T_{K_m(x)}(x)]$. However, these frequencies will not accurately reflect actual time averages of visits to a given box unless the length of a visit to that box is approximately independent of x. To remedy this, we add another condition.

Define $\phi: X_* \to \mathbf{R}^+$, $\phi(x) = T_1(x) - T_0(x)$. We view ϕ as measuring the length of a visit.

Definition 1.4. $\mathscr{A}$ is *δ-great* if it is both δ-good and satisfies
(IV) if $b_0(x) = b_0(y)$ then $|\phi(x) - \phi(y)| \leqq \delta\, \phi(x)$. ■

Remarks.

(i) Letting $\beta_i = \inf_{b_0(x)=i} \phi(x)$, we have from (iv): if $b_0(x) = i$ then $|\phi(x) - \beta_i| \leqq \delta\, \phi(x)$.

(ii) $T_{j+1}(x) - T_j(x) = \phi(\psi_{T_j(x)}(x))$ since $T_0(\psi_{T_j(x)}(x)) = 0$ and $T_1(\psi_{T_j(x)}(x)) = T_{j+1}(x) - T_j(x)$.

(iii) $|T_{j+1}(x) - T_j(x) - \beta_{b_j(x)}| \leqq \delta(T_{j+1}(x) - T_j(x))$ (by IV and (i) and (ii) above).

(iv) $\|\phi\| = \sup_{x\in X_*} \phi(x) < \infty$.

Proposition 1.5. *If for all $\delta > 0$ $\{\psi_t\}$ has a δ-great collection, then $\{\psi_t\}$ is uniquely ergodic.*

Proof. To show unique ergodicity we use the Time Average Criterion (1.1). So, let $h: X \to \mathbf{R}$ be continuous. We may assume that h is not identically 0 and for each x $0 \leqq h(x) \leqq 1$ (otherwise divide by $\|h\|$ and separate out positive and negative parts). We verify the condition of Lemma (1.2). So let $\varepsilon > 0$ and then choose δ such that $0 < \delta < \varepsilon$ and $|h(x) - h(y)| < \varepsilon$ when $d(x, y) < \delta$. Let $\mathscr{A}$ be δ-good. Let $X_\varepsilon = X_*$, $M_\varepsilon = \|\phi\| \cdot (M_*)$ and $r_\varepsilon = (\Sigma \alpha_i \beta_i r_i)/(\Sigma \beta_i r_i)$, where X_*,

M_* and r_i are as in Definition 1, β_i as in Remark (i) and $\alpha_i = \inf_{x \in A_i} h(x)$. We claim that X_ε, M_ε, and r_ε satisfy Lemma (1.2). So for $x \in X_\varepsilon$ we must produce a sequence $\{S_m(x)\}$. Let $S_m(x) = T_{K_m(x)}(x)$. ($K_m(x)$ as in Definition (1.3) III.)

For (a) of Lemma (1.2) first note that by Remark (ii) $T_{j+1}(x) - T_j(x) \leqq \|\phi\|$ for any j. Thus

$$0 \leqq S_{m+1}(x) - S_m(x) = \sum_{j=K_m(x)}^{K_{m+1}(x)-1} T_{j+1}(x) - T_j(x)$$

$$\leqq (K_{m+1}(x) - K_m(x))\|\phi\|$$

by III(a):

$$\leqq M_*\|\phi\| \leqq M_\varepsilon .$$

And

$$S_0(x) = T_{K_0(x)}(x) \leqq 0 < T_{K_1(x)}(x) = S_1(x) .$$

The proof of (b) (of Lemma (1.2)) is an approximation argument. To expedite matters we state a useful fact:

LEMMA 1.6. *If $a_i \geqq 0$, c_i, d_i, $e_i > 0$, $a_i \leqq e_i$ and $|c_i - d_i| \leqq \delta c_i$ for $i = 1, \cdots, m$, then*

$$\left| \frac{\sum a_i c_i}{\sum e_i c_i} - \frac{\sum a_i d_i}{\sum e_i d_i} \right| \leqq 2\delta .$$

PROOF. First show that $|u/v - w/y| \leqq 2\delta \max\{u/v, w/y\}$ if u, $w \geqq 0$, v, $y \geqq 0$, $|u - w| \leqq \delta u$, and $|v - y| \leqq \delta v$. Then reduce the lemma to this case. ∎

Now we estimate

$$\frac{1}{S_m(x) - S_0(x)} \int_{S_0(x)}^{S_m(x)} h \circ \psi_t(x)dt.$$

For brevity,

$$\sum_j = \sum_{j=K_0(x)}^{K_m(x)-1} ; \text{ and } \sum_i = \sum_{i \in \Lambda} .$$

(Λ is the index for $\mathscr{A}$.) Also we suppress dependence on x.

$$\text{(1)} \qquad \frac{1}{S_m - S_0} \int_{S_0}^{S_m} h \circ \psi_t(x)dt = \frac{\sum_j \int_{T_j}^{T_{j+1}} h \circ \psi_t(x)dt}{\sum_j (T_{j+1} - T_j)} .$$

Since $\mathscr{A}$ is δ-good, for $t \in [T_j, T_{j+1}]$, $\psi_t(x) \in A_{b_j}$, and since $\operatorname{diam} A_{b_j} < \delta$, we have $|h \circ \psi_t(x) - \alpha_{b_j}| \leqq \varepsilon$. Thus,

$$\left| \int_{T_j}^{T_{j+1}} h \circ \psi_t(x)dt - \alpha_{b_j}(T_{j+1} - T_j) \right| \leqq \varepsilon (T_{j+1} - T_j).$$

Thus,

$$(2)\qquad \left|\frac{\Sigma_j \int_{T_j}^{T_{j+1}} h\circ\psi_t(x)dt}{\Sigma_j(T_{j+1}-T_j)} - \frac{\Sigma_j\alpha_{b_j}\cdot(T_{j+1}-T_j)}{\Sigma_j(T_{j+1}-T_j)}\right| \leqq \varepsilon .$$

And by Lemma (1.6) and Remark (iii) (recall $\alpha_{b_j} \leqq \|h\| \leqq 1$):

$$(3)\qquad \left|\frac{\Sigma_j\alpha_{b_j}\cdot(T_{j+1}-T_j)}{\Sigma_j(T_{j+1}-T_j)} - \frac{\Sigma_j\alpha_{b_j}\cdot\beta_{b_j}}{\Sigma_j\beta_{b_j}}\right| \leqq 2\delta < 2\varepsilon ,$$

$$(4)\qquad \frac{\Sigma_j\alpha_{b_j}\cdot\beta_{b_j}}{\Sigma_j\beta_{b_j}} = \frac{\Sigma_i\alpha_i\beta_i\eta_{im}\cdot(K_m-K_0)}{\Sigma_i\beta_i\eta_{im}\cdot(K_m-K_0)} = \frac{\Sigma_i\alpha_i\beta_i\eta_{im}}{\Sigma_i\beta_i\eta_{im}} .$$

And by Lemma (1.6) and Condition III (b) of Definition (1.3):

$$(5)\qquad \left|\frac{\Sigma_i\alpha_i\beta_i\eta_{im}}{\Sigma_i\beta_i\eta_{im}} - \frac{\Sigma_i\alpha_i\beta_i r_i}{\Sigma_i\beta_i r_i}\right| \leqq 2\delta < 2\varepsilon .$$

Putting (1)–(5) together with the triangle inequality

$$\left|\frac{1}{S_m-S_0}\int_{S_0}^{S_m} h\circ\psi_t(x)dt - r_\varepsilon\right| < 5\varepsilon ,$$

completing the proof of Proposition 1.5. ■

2. Axiom A background

Let f: $M \to M$ be continuous. The non-wandering set

$$NW(f) = \{x \in M\colon U \cap (\textstyle\bigcup_{n>0} f^n U) \neq \varnothing \text{ for all neighborhoods } U \text{ of } x\}.$$

Let $f\colon M \to M$ be an Axiom A diffeomorphism [16] on a compact connected Riemannian manifold. This means

(a) Over $NW(f)$, the tangent bundle splits into a continuous sum of two invariant (under Df) subbundles, one of which (E^u) is expanded, the other (E^s) contracted.

(b) The periodic points of f are dense in $NW(f)$.

We take advantage of the stable manifold theory for f developed by Hirsch and Pugh [9]. For $x \in NW(f)$ and $\gamma > 0$, let

$$W^s_\gamma(x) = \{y \in M\colon d(f^n y, f^n k) \leqq \gamma \text{ for } n \geqq 0\}$$

$$W^s(x) = \{y \in M\colon \lim_{n\to+\infty} d(f^n y, f^n x) = 0\}$$

$$W^u_\gamma(x) = \{y \in M: d(f^n y, f^n x) \leqq \gamma \text{ for } n \leqq 0\}$$

$$W^u(x) = \{y \in M: \lim_{n \to -\infty} d(f^n y, f^n x) = 0\},$$

called the local stable, stable, local unstable, unstable manifold resp. (d is a Riemannian metric).

Fundamental Fact 2.1 [9]. There is a Riemannian metric d and constants $\gamma > 0$, $0 < \lambda < 1$ such that

(i) Each $W^s_\gamma(x)$ and $W^u_\gamma(x)$ are smoothly embedded disks tangent to E^s_x, E^u_x respectively.

(ii) for $n \geqq 0$ if $y, z \in W^s_\gamma(x)$ then $d(f^n y, f^n z) \leqq \lambda^n d(y, z)$,

for $n \leqq 0$ if $y, z \in W^u_\gamma(x)$ then $d(f^n y, f^n z) \leqq \lambda^n d(y, z)$.

(iii) (Canonical Coordinates) There exists $\alpha > 0$ such that if $x, y \in NW(f)$ and $d(x, y) < \alpha$ then $W^s_\gamma(y) \cap W^u_\gamma(x)$ is a single point $[y, x] \in NW(f)$; moreover the map

$$[\cdot, \cdot]: (W^u_\gamma(x) \cap NW(f)) x (W^s_\gamma(x) \cap NW(f)) \to NW(f)$$

is a homeomorphism onto a closed neighborhood (in $NW(f)$).

(iv) (Expansiveness) γ is an expansive constant, i.e., if $d(f^n x, f^n y) \leqq \gamma$ for all n then $x = y$; equivalently, given $\varepsilon > 0$ there exists $n = n(\varepsilon)$ such that if $d(f^i x, f^i y) \leqq \gamma$ for $|i| \leqq n$, then $d(x, y) < \varepsilon$.

This has some interesting consequences. For example, if $\nu \leqq \gamma$; then $W^u_\nu(x) = B_\nu(x) \cap W^u_\gamma(x)$ ($B_\nu(x) =$ ball of radius ν about x) and $W^u(x) = \bigcup_{n \geqq 0} f^n W^u_\nu(f^{-n} x)$. This makes $W^u(x)$ an immersed submanifold. Similarly for W^s. Note that W^u and W^s are transverse with complementary dimensions. Also, $f W^u(x) = W^u(fx)$, and if $y \in W^u(x)$, then $W^u(y) = W^u(x)$.

Notation. For $A, B \subset X$ sufficiently close, $[A, B] = \{[x, y]: x \in A, y \in B\}$; $W^u(x, A) = W^u_\gamma(x) \cap A$; $W^s(x, A) = W^s_\gamma(x) \cap A$.

Let $X \subset NW(f)$ be a basic set (i.e. closed, f-invariant and has a dense f-orbit). The study of $f|_X$ has benefited from R. Bowen's construction of Markov partitions [2]. Let $\delta > 0$ be small (much smaller than α or γ). A Markov partition of size δ is a collection of subsets of X, $\mathcal{M} = \{A_i\}_{i=1}^N$ such that

(i) $\mathcal{M}$ covers X and each diam $A_i < \delta$.

(ii) Each $A_i =$ Cl (int A_i).

(iii) (Rectangle Property) $x, y \in A_i$ implies $[x, y] \in A_i$ (δ is assumed small enough so that this makes sense).

(iv) For $i \neq j$, $A_i \cap A_j = \partial A_i \cap \partial A_j$.

(v) (Markov Property) if $x \in \text{int } A_i \cap f^{-1} \text{ int } A_j$ then $fW^s(x, A_i) \subset W^s(fx, A_j)$ and $fW^u(x, A_i) \supset W^u(fx, A_j)$.

Each A_i is called a rectangle. Note that the rectangles are allowed to meet on their boundaries; so this is not literally a partition.

FACT 2.2 [2]. Markov Partitions of arbitrary small size exist for $f|_X$. ■

Let
$$\partial^s A_i = \{x \in A_i : x \notin \text{int } W^u(x, A_i)\}, \quad \partial^s \mathcal{M} = \bigcup_{i=1}^{N} \partial^s A_i$$
$$\partial^u A_i = \{x \in A_i : x \notin \text{int } W^s(x, A_i)\}, \quad \partial^u \mathcal{M} = \bigcup_{i=1}^{N} \partial^u A_i$$

(the interiors above are taken relative to $W^u_\gamma(x)$ and $W^s_\gamma(x)$).

FACT 2.3 [2]. (i) Each $\partial A_i = \partial^s A_i \cup \partial^u A_i$.
(ii) $f\partial^s \mathcal{M} \subset \partial^s \mathcal{M}$.
(iii) $f\partial^u \mathcal{M} \supset \partial^u \mathcal{M}$. ■

NOTATION. $\partial \mathcal{M} = \bigcup_{i=1}^{N} \partial A_i = \partial^s \mathcal{M} \cup \partial^u \mathcal{M}$ and $\text{int}\, \mathcal{M} = \bigcup_{i=1}^{N} \text{int}\, A_i$.

The importance of a Markov Partition is that it allows one to use symbolic dynamics. We describe this as follows:

Let $C = (C_{ij})$ be the $N \times N$ matrix defined by

$$C_{ij} = \begin{cases} 1 & \text{if int } A_i \cap f^{-1} (\text{int } A_j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

C is the partition matrix for $\mathcal{M}$. Let Σ_C denote the set of all doubly infinite sequences $\underline{x} = (x_i)_{i=-\infty}^{+\infty}$ such that $a_{x_i x_{i+1}} = 1$ for each i. Let $\sigma : \Sigma_C \to \Sigma_C$ be the left shift $(\sigma(\underline{x}))_i = x_{i+1}$. (A subshift of finite type.)

FACT 2.4 (Symbolic Dynamics) [2], [3]).

(i) There is a continuous onto map $\pi : \Sigma_C \to X$ such that $f \circ \pi = \pi \circ \sigma$.

(ii) $\pi(\underline{x}) \in A_{x_0}$ and if $x \in A_i$, then there exists $\underline{x} \in \Sigma_C$ such that $x_0 = i$ and $\pi(\underline{x}) = x$. ■

We now describe the context we're interested in. An attractor X for an Axiom A diffeomorphism f is a basic set for which there exists a neighborhood U in M such that $\bigcap_{n \geq 0} f^n U = X$. A consequence of this is that for each $x \in X$, $W^u(x) \subset X$. Thus $\{W^u(x) : x \in X\}$ (the W^u foliation) partitions X into smooth immersed submanifolds with continuously varying tangent spaces $\{E^u_x\}$. For the remainder of the paper:

STANDING ASSUMPTIONS 2.5. (1) X is a connected attractor.

(2) $E^u|_X$ is orientable.

(3) $E^u|_X$ is 1-dimensional.

With assumptions (2) and (3) one may define a continuous flow $\{\psi_t\}$ on X whose orbits are the unstable manifolds $W^u(x)$ [18]. This is what we mean by saying that the W^u foliation admits a flow. We call the flow a W^u flow. Of course there are several such flows (with different parametrizations), but by definition they all have the same orbits. Hence if one is uniquely ergodic, then so is any other [11]. In view of this we could select any convenient W^u flow (e.g. parametrization by arc length) but that has no real advantage.

THEOREM 2.6. *Under the standing assumptions* (2.5) *the W^u flow is uniquely ergodic.*

Note that the connectedness assumption is required to eliminate some immediate counterexamples; for example, one could just create a new attractor from the union of two disjoint connected attractors. An important consequence of the connectedness assumption is that X is a C-dense basic set for f [4]. This implies

FACT 2.7 [4]. (i) Each $W^u(x)$ is dense in X.

(ii) $f|_X$ is topologically mixing (whence C^K has all positive entries for some K) (see [Remark following Proposition 30, 2] and [1.3, 5]). f will mean $f|_X$.

3. Proof of main result

We will use the condition of Proposition 1.5. Our boxes will be the rectangles of a Markov Partition. First, we will improve our picture of Markov Partitions using part of the standing assumption (2.5).

DEFINITION 3.1. A Markov Partition $\mathcal{M}$ is *U-connected* if each $W^u(x, A_i)$ is a (non-trivial, closed, connected) arc (non-trivial means that it isn't a single point).

LEMMA 3.2. *There exist U-connected Markov Partitions of arbitrarily small size.*

REMARK. D. Pixton helped us on this point.

PROOF. This relies heavily on the fact that X is an attractor and $W^u(x)$ is one-dimensional; for otherwise $W^u(x, A_i)$ will be badly disconnected.

First note that we can find $\mathcal{M}$ (of arbitrarily small size) such that each $W^u(x, A_i)$ has only finitely many components; to see this follow the construction of Markov Partitions [2] (there, on p. 731, under our assumptions we can choose C_i^0 connected; in Lemma 16 then, C_i will be connected and then in the end $W^u(x, E)$ will have a finite number of components). Now we just create a new Markov Partition by separating the components; namely, for each $A_i \in \mathcal{M}$, let $x_i \in A_i$ and $W_{i1}, W_{i2}, \cdots, W_{ik_i}$ be the components of $W^u(x_i, A_i)$. Note that since $A_i = \mathrm{Cl}(\mathrm{int}\, A_i)$ and A_i is homeomorphic to $W^u(x_i, A_i) \times W^s(x_i, A_i)$ via the canonical coordinate map (2.1) (iii), each W_{ij} must be a non-trivial closed connected arc. Let $\mathcal{M}' = \{A_{ij}\}$, where $A_{ij} = [W_{ij}, W^s(x_i, A_i)]$. This is a U-connected Markov Partition; we omit the details except to note that, for $x \in A_{ij}$, $W^u(x, A_{ij}) = [W_{ij}, x]$ and $W^s(x, A_{ij}) = W^s(x, A_i)$. Also $\mathcal{M}'$ is a refinement of $\mathcal{M}$.

∎

REMARK 3.3. For a U-connected Markov Partition, each $\partial^s A_i$ consists of exactly two stable fibers $W^s(x_i^1, A_i)$ and $W^s(x_i^2, A_i)$; each $W^u(x, A_i) \cap \partial^s A_i$ consists of exactly two points (the endpoints, $[x_i^1, x]$ and $[x_i^2, x]$) (see [Lemma 10, 2]; in fact $W^u(x, A_i) \cap \partial^s \mathcal{M}$ is a finite set, for each $W^u(x, A_i) \cap W^s(x_j^v, A_j)$ ($j = 1, \cdots N$, $v = 1, 2$) is either empty or a single point).

PROPOSITION 3.4. *Any U-connected Markov Partition of size δ is δ-good.*

PROOF. We must check (1.3) (I), (II), and (III). Of course (I) is immediate. To construct X_*, first let p be a periodic point in int $\mathcal{M}$ (although at the outset one knows only that the periodic points are dense in $NW(f)$, they must also be dense in X; see, for example, Smale's Spectral Decomposition [16]).

DEFINITION 3.5.

$$X_* = \bigcup_{i=0}^{k-1} W^u(f^i p) = \bigcup_{i=0}^{k-1} f^i W^u(p)$$

where k is the period of p. Note that X_* is both f- and ψ-invariant. It is dense in X since $W^u(p)$ is (2.7).

We will now check (II). First, note that $\{f^i p : i \in [0, k-1]\}$ (the f-orbit of p) is contained in int $\mathcal{M}$. For suppose that $f^i p \in \partial \mathcal{M} = \partial^u \mathcal{M} \cup \partial^s \mathcal{M}$ (see (2.3)).

$$\text{If } f^i p \in \partial^s \mathcal{M} \text{ then } p = f^{k-i} f^i p \in f^{k-i} \partial^s \mathcal{M} \subset \partial^s \mathcal{M}.$$

$$\text{If } f^i p \in \partial^u \mathcal{M} \text{ then } p = f^{-i} f^i p \in f^{-i} \partial^u \mathcal{M} \subset \partial^u \mathcal{M}.$$

In either case $p \in \partial \mathcal{M}$, a contradiction.

Next note that if $x \in X_*$ then $W^u(x) \cap \partial^u \mathcal{M} = \emptyset$. For this, it suffices to assume $x = p, fp, \cdots, f^{k-1}p$. So x is a periodic point of period k in $\operatorname{int} A$ for some $A \in \mathcal{M}$. So there exists $\nu > 0$ such that $B_\nu(x)$, the ball of radius ν about x, is contained in $\operatorname{int} A$. Thus $W^u_\nu(x) = B_\nu(x) \cap W^u_\gamma(x) \subset W^u(x, \operatorname{int} A)$. (See remarks after (2.1).) So,

$$W^u(x) = \bigcup_{n \geqq 0} f^{kn} W^u_\nu(x) = \bigcup_{n \geqq 0} f^{kn} W^u(x, \operatorname{int} A). \tag{6}$$

Since $(\operatorname{int} A) \cap \partial^u \mathcal{M} = \emptyset$, we have by (2.3) that for $n \geqq 0$ $W^u(x, \operatorname{int} A) \cap f^{-kn} \partial^u \mathcal{M} = \emptyset$. Thus $f^{kn}(W^u(x, \operatorname{int} A)) \cap \partial^u \mathcal{M} = \emptyset$. This and (6) above yield $W^u(x) \cap \partial^u \mathcal{M} = \emptyset$ as claimed.

Recalling notation in (1.3) (II), if $x \in X_*$, $B_+(x) = \{t \geqq 0: \psi_t(x)$ belongs to more than one $A_i\} = \{t \geqq 0: \psi_t(x) \in \partial^s \mathcal{M}\}$, the last equality holding since (i) rectangles meet only on their boundaries, (ii) $\partial \mathcal{M} = \partial^u \mathcal{M} \cup \partial^s \mathcal{M}$ and (iii) $W^u(x) \cap \partial^u \mathcal{M} = \emptyset$. Now $B_+(x) \neq \emptyset$ since one of the two points comprising $W^u(x, A_i) \cap \partial^s A_i$ must be in the positive semiorbit of x. Similarly, $B_-(x) \neq \emptyset$.

Note that $\partial^s \mathcal{M}$ consists of finitely many subsets of embedded disks transverse to $W^u(x)$ with complementary dimension. (Namely $\{W^s(x^1_i, A_i)\} \cup \{W^s(x^2_i, A_i)\}$ as in (3.3.) Since $t \to \psi_t(x)$ is a continuous parametrization of $W^u(x)$, it follows that $\{t: \psi_t(x) \in \partial^s \mathcal{M}\}$ is discrete. Thus $B_+(x)$ and $B_-(x)$ are discrete. This gives (IIa).

For II(b) recall that $B_+(x) \cup B_-(x) = \{T_j(x)\}$ (as in (1.3); if $T_j(x) < t < T_{j+1}(x)$, then $\psi_t(x) \in \operatorname{int} \mathcal{M}$; since $\{\psi_t(x): t \in (T_j(x), T_{j+1}(x))\}$ is connected, it and hence its closure is contained in some A_i, called $A_{b_j(x)}$. This is (IIb).

We now derive an easy consequence of $W^u(x) \cap \partial^u \mathcal{M} = \emptyset$ for $x \in X_*$, which shows that $W^u(x)$ has a unique representation as a union of $W^u(y, A_i)$'s.

For brevity let $W_j(x) = \{\psi_t(x): t \in [T_j(x), T_{j+1}(x)]\}$.

Lemma 3.6. (i) *If $y \in A_i \cap W^u(x)$ then $W^u(y, A_i) = W_j(x)$ for some j.*

(ii) $W^u(\psi_{T_j(x)}(x), A_{b_j(x)}) = W_j(x)$.

Proof. First note that if $W^u(y, A_i) \subset W^u(x)$, then $W^u(y, A_i) \cap \partial^s \mathcal{M} = W^u(y, A_i) \cap \partial^s A_i$ (the two endpoints of $W^u(y, A_i)$), for if $z \in W^u(y, A_i) - \partial^s A_i$ then $z \in \operatorname{int} W^u(y, A_i)$ (rel. $W^u_\gamma(y)$) and since $z \notin \partial^u \mathcal{M}$, $z \in \operatorname{int} W^s(y, A_i)$ (rel. $W^s_\gamma(y)$), whence z would have to be in $\operatorname{int} A_i$ (by canonical coordinates (2.1) (iii) and the rectangle property for A_i).

(i) Since $W^u(y, A_i)$ is a closed connected subarc of $W^u(x)$ and by the above $W^u(y, A_i) \cap \partial^s \mathcal{M} =$ its endpoints, we have by definition of $\{T_j(x)\}$ that $W^u(y, A_i) = W_j(x)$ for some j.

(ii) By (i), $W^u(\psi_{T_j(x)}(x), A_{b_j(x)}) = W_J(x)$ for some J. But by definition of $A_{b_j(x)}$ and the fact that $W^u_\gamma(\psi_{T_j(x)}(x))$ is an arc containing $\psi_{T_j(x)}(x)$ in its interior, there must exist $\varepsilon > 0$ such that

$$W^u(\psi_{T_j(x)}(x), A_{b_j(x)}) = A_{b_j(x)} \cap W^u_\gamma(\psi_{T_j(x)}(x)) \supset \{\psi_t(x): t \in [T_j(x), T_j(x)+\varepsilon]\}.$$

Thus $J = j$ as desired.

We will now verify (III) of (1.3). This is the crux of the matter. To get an idea of how each $W^u(x)$ is distributed we see how $f^n W^u(f^{-n}x, A_i)$ is distributed among the rectangles (after all, $f^n W^u(f^{-n}x, A_i)$ is a big chunk of $W^u(x)$). Our counting is done on the symbolic level in Lemmas (3.7) and (3.10). From here through (3.10) there is actually no need to assume that any of the points we consider belong to X_*.

Let $S(n,l)$ denote the set of all sequences $a = a_{-n}\, a_{-n+1} \cdots a_0$ such that $a_{-n} = l$ and $C_{a_j a_{j+1}} = 1$ for j: $-n \leqq j \leqq -1$ (where $C = (C_{ij})$ is the partition matrix for $\mathcal{M}$).

LEMMA 3.7. *Let $n > 0$ and $z \in A_l \in \mathcal{M}$. For each $a \in S(n,l)$ there exists $z^a \in A_{a_0}$ such that*

(i) $f^n W^u(z, A_l) = \bigcup_{a \in S(n,l)} W^u(z^a, A_{a_0})$,

(ii) *for $a, a' \in S(n,l)$ $a \neq a'$, $W^u(z^a, A_{a_0})$ is distinct from $W^u(z^{a'}, A_{a_0})$. In fact, they meet only in their endpoints, if at all.*

For this we will need two preliminary facts.

For $\underline{x} \in \Sigma_C$ let $U(\underline{x}) = \{\underline{y} \in \Sigma_C : y_i = x_i, i \leqq 0\}$ and $S(\underline{x}) = \{\underline{y} \in \Sigma_C : y_i = x_i$ for $i \geqq 0\}$. Note that if $x_0 = y_0$ then $U(\underline{x}) \cap S(\underline{y})$ is a single point.

SUBLEMMA 3.8. $\pi(U(\underline{x})) = W^u(\pi(\underline{x}), A_{x_0})$.

PROOF. By (2.4), for any n, $f^n \circ \pi^n(\underline{x}) = \pi \circ \sigma^n(\underline{x}) \in A_{x_n}$. Thus, if $\underline{y} \in U(\underline{x})$ then for $n \leqq 0$, $f^n \circ \pi(\underline{y}) \in A_{y_n} = A_{x_n}$. So for $n \leqq 0$, $d(f^n \circ \pi(\underline{x}), f^n \circ \pi(\underline{y})) \leqq \delta < \gamma$ (recall that the size δ was assumed to be $< \gamma$). Thus $\pi(\underline{y}) \in W^u_\gamma(\pi(\underline{x}))$. And by (2.4) (ii) $\pi(\underline{y}) \in A_{y_0} = A_{x_0}$. Thus

$$\pi(\underline{y}) \in W^u_\gamma(\pi(\underline{x})) \cap A_{x_0} = W^u(\pi(\underline{x}), A_{x_0}).$$

For the other half, let $y \in W^u(\pi(\underline{x}), A_{x_0})$. By (2.4) (ii) we can find $\underline{y} \in \Sigma_C$ such that $y_0 = x_0$ and $\pi(\underline{y}) = y$. Let $\underline{y}' = U(\underline{x}) \cap S(\underline{y})$. Now

$$\pi(\underline{y}') \in \pi(U(\underline{x})) \cap \pi(S(\underline{y})) \subset W^u(\pi(\underline{x}), A_{x_0}) \cap W^s(\pi(\underline{y}), A_{x_0})$$

(since $\pi(U(\underline{x})) \subset W^u(\pi(\underline{x}), A_{x_0})$ as proved above and $\pi(S(\underline{y})) \subset W^s(\pi(\underline{y}), A_{x_0})$ as proved analogously). But then $y = [y, \pi(\underline{x})] = [\pi(\underline{y}), \pi(\underline{x})] = \pi(\underline{y}')$ (the first equality since y was chosen in $W^u(\pi(\underline{x}), A_{x_0})$. Since $\underline{y}' \in U(\underline{x})$ we have $y = \pi(\underline{y}') \in \pi(U(\underline{x}))$ as desired. ∎

SUBLEMMA 3.9. *Let $\underline{x}$, $\underline{y} \in \Sigma_C$ with $\pi(\underline{x}) = \pi(\underline{y})$. If $x_{-n} = y_{-n}$ for some $n > 0$, but $x_{-j} \neq y_{-j}$ for some j: $0 \leqq j < n$, then $\pi(\underline{x}) \in \partial^s \mathcal{M}$.*

PROOF. Choose $n > 0$ such that $x_{-n} = y_{-n}$, but $x_{-n+1} \neq y_{-n+1}$. By (2.4), $f^{-n}(\pi(\underline{x})) \in A_{x_{-n}} \cap A_{y_{-n}} = A_{x_{-n}}$ and $f^{-n+1}(\pi(\underline{x})) \in A_{x_{-n+1}} \cap A_{y_{-n+1}}$. Then by [Dual Version of Lemma 6,3], $f^{-n+1}(\pi(\underline{x})) \in \partial^s \mathcal{M}$. Thus $\pi(\underline{x}) = f^{n-1} f^{-n+1}(\pi(\underline{x})) \in f^{n-1} \partial^s \mathcal{M} \subset \partial^s \mathcal{M}$, since $n - 1 \geqq 0$. ∎

PROOF OF LEMMA 3.7. Choose $\underline{z} \in \Sigma_C$ such that $z_0 = l$ and $\pi(\underline{z}) = z$. By (3.8), $W^u(z, A_l) = \pi(U(\underline{z}))$. Thus

$$f^n W^u(z, A_l) = f^n \circ \pi(U(\underline{z})) = \pi \circ \sigma^n(U(\underline{z})). \tag{7}$$

For $a \in S(n, l)$, let $\underline{z}^a \in \Sigma_C$ be defined by

$$z_j^a = a_j \qquad \text{for} \quad -n \leqq j \leqq 0,$$

$$z_j^a = z_{j+n} \qquad \text{for} \quad j < -n,$$

and for $j > 0$ choose z_j^a in any way so that $\underline{z}^a \in \Sigma_C$; this is possible since there exists a_1 such that $C_{a_0 a_1} = 1$, etc. (See the definition of C preceding (2.4).) Then $\sigma^n(U(\underline{z})) = \bigcup_{a \in S(n,l)} U(\underline{z}^a)$. So this, (7), and (3.8) imply

$$f^n W^u(z, A_l) = \bigcup_{a \in S(n,l)} \pi(U(\underline{z}^a)) = \bigcup_{a \in S(n,l)} W^u(\pi(\underline{z}^a), A_{a_0}).$$

This gives (i), with $z^a = \pi(\underline{z}^a)$.

For (ii) note that if $z \in W^u(\pi(\underline{z}^a), A_{a_0}) \cap W^u(\pi(\underline{z}^{a'}), A_{a'_0})$, a, $a' \in S(n, l)$, $a \neq a'$, then $z = \pi(\underline{x}) = \pi(\underline{y})$ for some $\underline{x}$, $\underline{y}$ satisfying the hypothesis of (3.9). Thus $z \in \partial^s \mathcal{M}$. Since $\partial^s \mathcal{M}$ meets $W^u(\pi(\underline{z}^a), A_{a_0})$ in a finite set (see (3.3)), $W^u(\pi(\underline{z}^a), A_{a_0})$ and $W^u(\pi(\underline{z}^{a'}, A_{a'_0})$ must be different. ∎

The following version of the Perron-Frobenius theorem is the technical tool we use to do our counting. Let $C_{li}^{(n)}$ denote the (l, i)th entry of C^n.

LEMMA 3.10. *There exist $r_1, \cdots, r_N > 0$ such that $\sum_{i=1}^N r_i = 1$ and for all l and i in $\{1, \cdots, N\}$*

$$\lim_{n \to \infty} \frac{C_{li}^{(n)}}{\sum_{u=1}^N C_{lu}^{(n)}} = r_i.$$

PROOF. First note that there is an integer K such that C^k has all positive entries (2.7) (ii).

Let C_* be the transpose of C. Now the Perron-Frobenius Theorem [1] asserts that C_* has a unique positive eigenvalue λ with eigenvector $r = (r_1, \cdots, r_N)$, $r_i \geqq 0$. We may assume $\Sigma_{i=1}^N r_i = 1$. Since C_*^K has all positive entries, each $r_i > 0$ and for each $v = (v_1, \cdots, v_N) \neq 0$ with $v_i \geqq 0$, there is a scalar $d_v > 0$ such that $\lim_{n\to\infty}(C_*^{Kn}v/\lambda^{Kn}) = d_v r$ (component-wise). This is essentially [Theorem 4, p. 292, 1]. Applying C_* to both sides of this equation K times, we get $\lim_{n\to\infty}(C_*^n v/\lambda^n) = d_v r$. Now let $\|v\| = \Sigma_{u=1}^N v_u$ with v as above. Then $\lim_{n\to\infty}(\|C_*^n v\|/\lambda^n) = d_v\|r\| = d_v$. Thus

$$\lim_{n\to\infty}\frac{C_*^n v}{\|C_*^n v\|} = \lim_{n\to\infty}\frac{C_*^n v/\lambda^n}{\|C_*^n v\|/\lambda^n} = r.$$

Apply this to the vector $v^l = (v_1, \cdots, v_N)$, where $v_l = 1$ and $v_j = 0$ for $j \neq l$. This gives

$$\lim_{n\to\infty}\frac{C_{li}^{(n)}}{\Sigma_{u=1}^N C_{lu}^{(n)}} = \lim_{n\to\infty}\frac{(C_*^n v^l)_i}{\|C_*^n v\|} = r_i.$$

■

Now let $r_1, \cdots, r_N$ be as in (3.10). Fix $n > 0$ an even integer such that for all l and i

$$\left|\frac{C_{li}^{(n)}}{\Sigma_{u=1}^N C_{lu}^{(n)}} - r_i\right| < \delta r_i. \tag{8}$$

Let $M_* = \max_{1\leq l\leq N}\Sigma_{u=1}^N C_{lu}^{(n)}$. We claim that M_* and the r_i satisfy (1.3) (III). S given $x \in X_*$, we must produce $\{K_m(x)\}$ satisfying (IIIa) and (IIIb).

Now recall the definition (3.5) of X_*. Since X_* is f-invariant, $f^{-n}x \in X$ Consider $W_q(f^{-n}x)$ (as preceding (3.6)) for some integer q. As in (3.7), there exist z^a, $a \in S(n,l)$ (where $l = b_q(f^{-n}x)$ and $z = \psi_{T_j(f^{-n}x)}(f^{-n}x)$ such that

$$\bigcup_{a\in S(n,l)} W^u(z^a, A_{a_0}) = f^n W_q(f^{-n}x) \subset W^u(x).$$

By (3.6) (i), for each $a \in S(n,l)$ there is a j such that $W^u(z^a, A_{a_0}) = W_j(x)$. This defines a map P: $S(n,l) \to Z$ by $P(a) = j$. Now note that P is $1-1$ and maps onto an interval of integers $[K_q, V_q]$. The former follows from (3.7) (ii) and the latter from the connectedness of $f^n W_q(f^{-n}x)$ (recall that $W_q(f^{-n}x)$ is connected by U-connectedness). In fact $V_q = K_{q+1} - 1$ since $f^n W_q(f^{-n}x)$ and $f^n W_{q+1}(f^{-n}x)$ have exactly one point in common and f^n preserves the orientation of the flow (recall that n is even and X connected). Since $W^u(z^a, A_{a_0}) = W_{P(a)}$, we have $a_0 = b_{P(a)}$. Thus, since P is $1-1$ onto $[K_q, K_{q+1}-1]$,

$$\#\{a \in S(n,l): a_0 = i\} = \#\{j \in [K_q, K_{q+1}-1]: b_j(x) = i\}$$

and the former, by a computation, is $C_{li}^{(n)}$, the (l,i)th entry of C^n. Thus

$$K_{q+1} - K_q = \sum_{u=1}^{N} \#\{j \in [K_q, K_{q+1}-1]: b_j(x) = u\} \tag{9}$$

$$= \sum_{u=1}^{N} C_{lu}^{(n)}.$$

And so

$$\frac{\#\{j \in [K_q, K_{q+1}-1]: b_j(x) = i\}}{K_{q+1} - K_q} = \frac{C_{li}^{(n)}}{\sum_{u=1}^{N} C_{lu}^{(n)}} \tag{10}$$

and by the way n was chosen (8) this quantity is within δr_i of r_i.

Now define $K_m(x) = K_m$. Then with $\eta_{im}(x)$ as in (1.3),

$$\eta_{im}(x) = \frac{\sum_{q=0}^{m-1} \#\{j \in [K_q, K_{q+1}-1]: b_j(x) = i\}}{\sum_{q=0}^{m-1}(K_{q+1} - K_q)};$$

that $\eta_{im}(x)$ is within δr_i of r_i follows from (10) and:

Arithmetic Inequality. If $d_q \geqq 0, e_q > 0$ for $q = 0, \cdots, m-1$, then

$$\min \frac{d_q}{e_q} \leqq \frac{\Sigma d_q}{\Sigma e_q} \leqq \max \frac{d_q}{e_q}.$$

Proof. The middle expression is a weighted average of d_q/e_q. ■

This gives III(b) of (1.3).

For III(a): by (9), $K_{q+1} - K_q = \sum_{u=1}^{N} C_{lu}^{(n)}$. Thus $0 < K_{q+1} - K_q \leqq M_*$. As for K_0, note that $f^{-n}x \in W_0(f^{-n}x)$; thus

$$x \in f^n W_0(f^{-n}x) = \bigcup_{j=K_0}^{K_1} W_j(x).$$

(The latter equality comes from the definition of K_0 and K_1.) So $K_0 \leqq 0 < K_1$. This completes the proof of (3.4). ■

To get a δ-great collection we refine $\mathcal{M}$. What will make this work is the following:

Definition 3.11. For $\mathcal{M}$ U-connected and $A_i \in \mathcal{M}$ define $L_i: A_i \to \mathbf{R}^+$ by $L_i(x) = t_2 - t_1$, where $W^u(x, A_i) = \{\psi_t(x): t \in [t_1, t_2]\}$. ■

REMARK 3.12. L_i is continuous. For this let $\partial^s A_i = W^s(x_i^1, A_i) \cup W^s(x_i^2, A_i)$ (as in (3.3)). If $x \in A_i$, then $[x_i^v, x] = \psi_{t_x^v}(x)$ for some t_x^v, $v = 1,2$. Then $L_i(x) = |t_x^1 - t_x^2|$. That the map $x \to t_x^v$ is continuous follows from the fact that each $W^s(x_i^v, A_i)$, being transverse to the flow, is a local cross-section [13]. ■

Let $\mathcal{M}$ be a U-connected Markov Partition of size δ. For $a = a_0 \cdots a_n$, let $A_a = \bigcap_{j=0}^n f^j A_{a_j}$. Let

$$\mathcal{M}_n = \{A_a : \bigcap_{j=0}^{n} f^j(\operatorname{int} A_{a_j}) \neq \emptyset\} \ (= \mathcal{M} \vee f\mathcal{M} \vee \cdots \vee f^n\mathcal{M}).$$

LEMMA 3.13. (i) *$\mathcal{M}_n$ is a U-connected Markov Partition of size δ.*

(ii) *For $A_a \in \mathcal{M}_n$, $W^u(x, A_a) = W^u(x, A_{a_0})$.*

PROOF. For $n = 2$, repeatedly apply [Lemma 26, 2], and then proceed by induction. ■

PROPOSITION 3.14. *For sufficiently large n, $\mathcal{M}_n$ is δ-great.*

PROOF. By (3.13) and (3.4), $\mathcal{M}_n$ is δ-good. So we must verify (IV) of (1.4), for $\mathcal{A} = \mathcal{M}_n$, some large n. To see how large: first let $L = \min_{i, x \in A_i} L_i(x) > 0$ (L_i as in (3.11)). By (3.12) there exists $\varepsilon > 0$ such that if $x, y \in A_i$ and $d(x,y) < \varepsilon$, then $|L_i(x) - L_i(y)| \leqq \delta L$. By expansiveness (2.1) (iv) there exists n such that if $d(f^j x, f^j y) \leqq \gamma$ for $|j| \leqq n$, then $d(x,y) \leqq \varepsilon$. For this n, we claim that $\mathcal{M}_n$ will do, as follows.

First note that

(11) $$\text{if } x, y \in A_a \in \mathcal{M}_n, \text{ then } |L_{a_0}(x) - L_{a_0}(y)| \leqq \delta L.$$

To see this, first note that $[x,y] \in A_a$ (rectangle property), whence for $0 \leqq j \leqq n$, both $f^{-j}x, f^{-j}[x,y] \in A_{a_j}$, whence $d(f^{-j}x, f^{-j}[x,y]) \leqq \delta < \gamma$. Also, since $[x,y] \in W^s(x, A_a) \subset W^s_\gamma(x)$, we have $d(f^j x, f^j[x,y]) \leqq \gamma$ for all $j \geqq 0$. By the way n was chosen, $d(x,[x,y]) < \varepsilon$ and thus $|L_{a_0}(x) - L_{a_0}([x,y])| \leqq \delta L$. But L_{a_0} is constant on $W^u(y, A_{a_0})$. So $L_{a_0}(y) = L_{a_0}([x,y])$. This gives (11).

Let X^*, $\{b_j(x)\}$, $\{T_j(x)\}$ be as in (1.3) for $\mathcal{A} = \mathcal{M}_n$. By (3.6) (ii) applied to $\mathcal{M}_n$, recalling $T_0(x) \leqq 0 < T_1(x)$, we get $W^u(x, A_{b_0(x)}) = \{\psi_t(x) : t \in [T_0(x), T_1(x)]\}$. By (3.13) (ii), $W^u(x, A_{b_0(x)}) = W^u(x, A_{a_0})$, where $b_0(x) = a_0 \cdots a_n$. Putting these together and recalling (3.11), we get $L_{a_0}(x) = T_1(x) - T_0(x)$. And by the definition of ϕ in (1.4), this is also $\phi(x)$. Thus, if $b_0(x) = b_0(y) = a_0 \cdots a_n$, then by (11) we have

$$|\phi(x) - \phi(y)| = |L_{a_0}(x) - L_{a_0}(y)| \leqq \delta L \leqq \delta L_{a_0}(x) = \delta\phi(x).$$

So $\mathcal{M}_n$ is δ-great. ■

Propositions (1.5) and (3.14) complete the proof of (2.6).

4. Equicontinuity

A flow is called minimal if every orbit is dense. Although limiting time averages (1.1) are invariant along orbits, they are not necessarily continuous. So a minimal flow need not be uniquely ergodic (see [13] for an example of this). However, if a minimal flow has the additional property that once two points start close, they stay close for all time, then it will be uniquely ergodic.

DEFINITION 4.1. A flow $\{\psi_t\}$ is *equicontinuous* if $\{\psi_t\}$ form an equicontinuous family of homeomorphisms. ■

Note that, in contrast to both unique ergodicity and minimality, this concept depends not just on the orbits but also on the parametrization of the flow. For example, one could reparametrize a rotation of the torus (which is equicontinuous) to get a non-equicontinuous flow. (View it as a suspension of a rotation on the circle.) Nevertheless,

LEMMA 4.2. *Every minimal equicontinuous flow is uniquely ergodic.*

PROOF. This is a direct modification of the corresponding result for a single homeomorphism [1.2, 7]. ■

In view of this one might ask if W^u flows are more than uniquely ergodic, i.e., are they equicontinuous? Since equicontinuity depends on the parametrization, the question is more properly formulated: Under (2.5), do W^u foliations admit equicontinuous W^u flows? Well, some don't. In fact,

(4.3) *Any compact metric space which supports a minimal equicontinuous flow is an abelian topological group.*

(To see this as in [p. 131, 17], define a multiplication on a single orbit $\{\psi_t(x)\}$ by $\psi_t(x) * \psi_s(x) = \psi_{t+s}(x)$; now extend this to the whole space by equicontinuity of $\{\psi_t\}$ and denseness of $\{\psi_t(x)\}$.) Now it is known that some 1-dimensional attractors satisfying (2.5) are not homogeneous [21]—hence not topological groups. Since W^u flows are minimal (2.7), we have by (4.3) that in these cases they couldn't be equicontinuous. We give a specific example of this at the end of the section.

However, a large class of W^u foliations do admit equicontinuous W^u flows. Let f be a codimension one Anosov diffeomorphism (with $\dim E^u = 1$) [6]. This is just the case of assumptions (2.5) with $X = M$. (That $NW(f) = M$ follows from Newhouse [12]; that E^u is orientable follows from (4.5) below.)

THEOREM 4.4. *The W^u foliation of a codimension one Anosov diffeomorphism (with $\dim E^u = 1$) admits an equicontinuous W^u flow.*

PROOF. First note that if g and f are topologically conjugate (i.e., there is a homeomorphism h such that $g = h \circ f \circ h^{-1}$) then the conjugating map h sends the unstable manifolds of f onto those of g. Thus if $\{\phi_t\}$ is a W^u flow for the W^u foliation of f, then $\psi_t \equiv h \circ \phi_t \circ h^{-1}$ is a W^u flow for g. Morever, if $\{\phi_t\}$ is equicontinuous, then so is $\{\psi_t\}$.

Thus, given a codimension one Anosov diffeomorphism g it suffices to find an f, whose W^u foliation admits an equicontinuous W^u flow, and is topologically conjugate to g. For this,

THEOREM (Franks [6], Newhouse [12]) 4.5. *Every codimension one Anosov diffeomorphism is topologically conjugate to a hyperbolic toral automorphism f (i.e., a map of the torus, $T^n = \mathbf{R}^n / Z^n$ induced by a linear transformation on $\mathbf{R}^n$ given by a matrix A with integer entries, no eigenvalues on the unit circle and* $\det A = \pm 1$).

Now the unstable manifolds for f are the images (via natural projection $P: \mathbf{R}^n \to T^n$) of a 1-dimensional subspace E (of $\mathbf{R}^n$) and its translates. $P(E)$ is a one-parameter subgroup of T^n (with the inherited group structure), and the W^u flow $\{\phi_t\}$, defined by translation by the subgroup $P(E)$, is equicontinuous since T^n has a translation invariant metric. As noted before, this implies the same for $\{\psi_t\}$. ■

REMARK 4.6. (4.4) and (4.2) give another proof of (2.6) in this case.

For the remainder of this section, we sketch an example of a one dimensional attractor whose W^u foliation satisfies (2.5) but does not admit an equicontinuous W^u flow. The model for these attractors is the generalized solenoid ([19]): the space X is the inverse limit system $K \overset{g}{\leftarrow} K \overset{g}{\leftarrow} K \overset{g}{\leftarrow} K \cdots$, where K is a branched 1-manifold and $g: K \to K$ satisfies

(4.7) (i) *g is an expanding immersion.*

(ii) $NW(g) = K$.

(iii) *Every point of K has a neighborhood whose image under g is an arc.*

(More liberal conditions are given in [20].) The shift map $f: X \to X$ $(f(x_0, x_1, x_2, \cdots) = (gx_0, x_0, x_1, x_2 \cdots))$ can be embedded as an attractor for an Axiom A diffeomorphism [19], and $E^u|_X$ is one dimensional. If K is connected, so is X. If K is orientable, then an orientation on K determines one on $E^u|_X$, so that the projection (from $W^u(x)$ to K via the zeroth coordinate map) is orientation-preserving. So under these two assumptions, $f: X \to X$ satisfies (2.5).

Roughly, the foliation may fail to admit an equicontinuous W^u flow because the paths traced by projections of two unstable manifolds are likely to eventually diverge at the branch points in K. To illustrate, let K be the topological wedge of two circles e_1 and e_2 with the (positive) orientation indicated in Fig. 1. Our map $g: K \to K$ is an expanding, orientation-preserving immersion which fixes p, expands and wraps e_1 first around e_1, then e_2, then e_1 again, and wraps e_2 around e_1 twice. So $g^{-1}(p)$ consists of p and three other points q, r, s as indicated in Fig. 1. So g maps the positively oriented arcs determined by these points as in Fig. 2. Note that g satisfies (4.7) (for (ii) observe that any subarc is eventually mapped onto all of K).

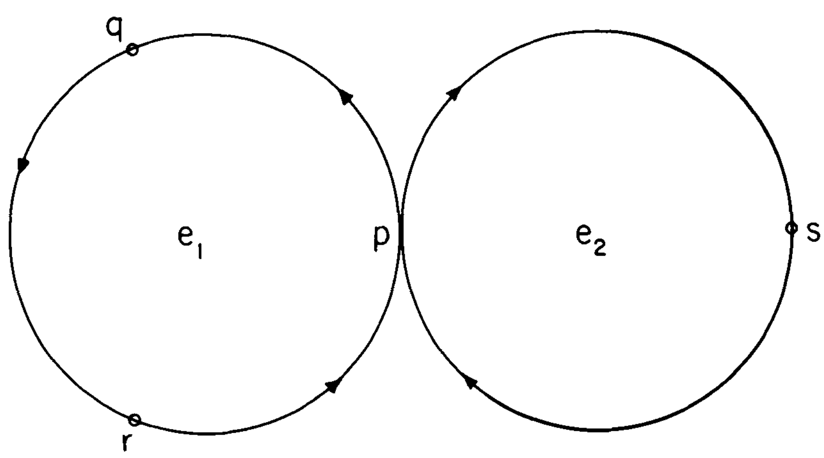

Fig. 1

$$\begin{aligned} g: [pq] &\to e_1 \\ [qr] &\to e_2 \\ [rp] &\to e_1 \\ [ps] &\to e_1 \\ [sp] &\to e_1 \end{aligned}$$

Fig. 2

We make some

DEFINITIONS 4.8. (i) By a *positive path* we mean a continuous positively oriented (locally 1–1) map $\sigma: [0, T] \to K$, $0 < T < +\infty$ (or $\sigma: [0, +\infty) \to K$).

(ii) Since σ follows the orientation of K it can't double back, and so if $\sigma(0) = p = \sigma(T)$ (or just $\sigma(0) = p$ if $\sigma: [0, +\infty) \to K$), σ determines an (ordered) word $\underline{\sigma}$ (possibly of infinite length) in the symbols e_1 and e_2.

(iii) An *initial segment* (I.S.) of a positive path σ is another positive path α such that $\alpha = \sigma \circ \beta$ for some (orientation preserving) homeomorphism β from the domain of α into the domain of σ, fixing 0.

(iv) If $\{\psi_t\}$ is a W^u flow on X and $z \in X$, define $\psi_z: [0, +\infty) \to K$ by $\psi_z(t) = \pi_0 \circ \psi_t(z)$, where $\pi_0: X \to K$ is projection onto the zeroth coordinate. ■

Note that we may assume that each ψ_z is a positive path.

REMARK 4.9. If α is an I.S. of ψ_z, then $g^n \circ \alpha$ is an I.S. of $\psi_{f^n z}$.

For this, use $g^n \circ \pi_0 = \pi_0 \circ f^n$ and the fact that f preserves the orientation of the flow (the latter because g preserves the orientation of K). ■

Now we go by contradiction. Suppose that $\{\psi_t\}$ is an equicontinuous W^u flow. Then $\{\pi_0 \circ \psi_t\}$ is an equicontinuous family. So if $x, y \in X$ with $\pi_0(x) = p = \pi_0(y)$ and x and y are sufficiently close, then $\psi_x = \psi_y$ (look at K to see that that's the only way ψ_x and ψ_y can stay close together for all $t \geqq 0$). Now let $x = (x_i)_{i \geqq 0}$, $y^{(n)} = (y_i^{(n)})_{i \geqq 0}$, where

$$x_i = p \text{ for all } i \geqq 0$$

$$y_i^{(n)} = \begin{cases} p & \text{for } i = 0, \cdots, n \\ q & \text{for } i = n+1 \\ \text{anything} & \text{for } i > n+1, \end{cases}$$

provided $y_i^{(n)} \in X$ (such points do exist). Since $y^{(n)}$ converges to x, we have that for sufficiently large n (by the remarks above), letting $y = y^{(n)}$,

$$\underline{\psi_x} = \underline{\psi_y}. \tag{12}$$

Let α be a positive path which traverses $[qr]$ once (see Fig. 1). Then α is an I.S. of $\psi_{f^{-(n+1)}y}$. Letting $\alpha_2 = g \circ \alpha$, we have by (4.9) that

$$g^n \circ \alpha_2 \text{ is an I.S. of } \psi_y. \tag{13}$$

Note that α_2 is a positive path, traversing e_2 once. (i.e., $\underline{\alpha_2} = e_2$).

Now let α_1 be a positive path, traversing e_1 once. Either α_1 or α_2 is an I.S. of ψ_x, whence by (4.9) either $g \circ \alpha_1$ or $g \circ \alpha_2$ is an I.S. of $\psi_{fx} = \psi_x$. But α_1 is an I.S. of both $g \circ \alpha_1$, and $g \circ \alpha_2$; so α_1 must be an I.S. of ψ_x. Applying (4.9) again, we get

$$g^n \circ \alpha_1 \text{ is an I.S. of } \psi_x. \tag{14}$$

Putting (12), (13), and (14) together, we get that both words $g^n \circ \alpha_1$ and $g^n \circ \alpha_2$

are initial blocks of a common word $\underline{\psi_x} = \underline{\psi_y}$. Now note that $\underline{g \circ \alpha_1} = \underline{\alpha_1}\ \underline{\alpha_2}\ \underline{\alpha_1}$ $(= e_1\ e_2\ e_1)$ and $\underline{g \circ \alpha_2} = \underline{\alpha_1}\ \underline{\alpha_1}$ $(= e_1\ e_1)$. Thus

$$\underline{g^n \circ \alpha_1} = \underline{(g^{n-1} \circ \alpha_1)}\ \underline{(g^{n-1} \circ \alpha_2)}\ \underline{(g^{n-1} \circ \alpha_1)} \text{ and}$$

$$\underline{g^n \circ \alpha_2} = \underline{(g^{n-1} \circ \alpha_1)}\ \underline{(g^{n-1} \circ \alpha_1)}.$$

Comparing these we see that since $\underline{g^n \circ \alpha_1}$ and $\underline{g^n \circ \alpha_2}$ are initial blocks of a common word, so are $\underline{g^{n-1} \circ \alpha_2}$ and $\underline{g^{n-1} \circ \alpha_1}$. Continuing inductively we find that $\underline{\alpha_1} = e_1$ and $\underline{\alpha_2} = e_2$ are initial blocks of a common word. Ridiculous!

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